Quantitative Homogenization of Interacting Particle Systems

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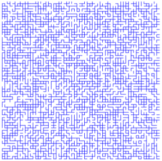
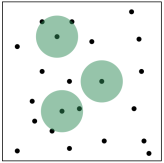
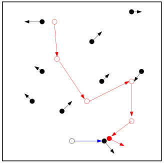
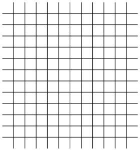
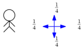
Universality of Brownian Motion

- Brownian universality: the limit random variable has the law of Brownian motion despite of the exact law of microscopic behaviors.
- Motivation: Go beyond the sum of independent random variables. Do these results (CLT, local CLT, invariance principle) also hold for other models (random walk in random environments, particles with interactions, hard-sphere model with collisions, etc)?

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Main Results

Random Conductance Model

- Sample i.i.d. random conductance $\{\mathbf{a}(e)\}_{e\in E_d}$.
- Let $(Y_t)_{t\geqslant 0}$ be a continuous-time Markov jump process starting from y, with an associated generator either
  - variable speed random walk VSRW

$$L_V^{\mathbf{a}} u(x) := \sum_{z\sim x} \mathbf{a}(\{x,z\}) \left(u(z) - u(x)\right);$$

  - constant speed random walk CSRW

$$L_C^{\mathbf{a}} u(x) := \sum_{z\sim x} \frac{\mathbf{a}(\{x,z\})}{\pi(x)} \left(u(z) - u(x)\right),$$

  with $\pi(x) := \sum_{z\sim x} \mathbf{a}(\{x,z\})$.

IP for RCM

Theorem (Invariance Principle)

When $0 < c \leqslant \mathbf{a} \leqslant C < \infty$, for almost every realization of $\{\mathbf{a}(e)\}_{e \in E_d}$, the scaling limit of VSRW or CSRW is Brownian motion.

$$\left(\frac{1}{\sqrt{n}} Y_{nt}\right)_{t \geqslant 0} \Rightarrow (\bar{\sigma} B_t)_{t \geqslant 0}.$$

- The condition $\mathbf{a} \in [c, C]$, i.i.d. is natural, but can be relaxed to other cases including
  - stationary ergodic environment;
  - supercritical percolation model;
  - degenerated i.i.d. conductance;
  - long-range jump and Levy-type limit;
  - ...
- Pioneer works by Sidoravicius, Sznitman, Biskup, Berger, Mathieu, Piatnitski, Barlow, Hambly, Kumagai, Bella, Schäffner, Chen, Chen, Wang, etc.

Outline

Homogenization Theory

- Elliptic Dirichlet problem with random, symmetric, $\mathbb{Z}^d$-stationary and ergodic coefficient in a large domain

$$\begin{cases} -\nabla \cdot (\mathbf{a}\nabla u) = f & \text{in } Q_r, \\ u = g & \text{on } \partial Q_r. \end{cases}$$

- For very large r, the solution can be approximated by the homogenized solution $\bar{u}$ for

$$\begin{cases} -\nabla \cdot (\bar{\mathbf{a}}\nabla \bar{u}) = f & \text{in } Q_r, \\ \bar{u} = g & \text{on } \partial Q_r, \end{cases}$$

where $\bar{\mathbf{a}} \in \mathbb{R}^{d\times d}$ is the (deterministic) effective coefficient.

- Approximation in the sense $u \simeq \bar{u}$ in L^2, and

$$\textit{gradient} : \nabla u \simeq \nabla \bar{u}, \qquad \textit{flux} : \mathbf{a}\nabla u \simeq \bar{\mathbf{a}}\nabla \bar{u} \text{ in } H^{-1}.$$

RCM and Homogenization

- Take $f = 0$ in both equations.
- Probabilistic representation is $\mathbb{E}^{\mathbf{a}}[g(Y_\tau)]$ for the hitting time τ of the boundary, which should be very close to that of "$\mathbb{E}[g(\bar{\sigma} B_\tau)]$", with $\bar{\mathbf{a}} = \frac{1}{2}\bar{\sigma}^2$.
- Usually $\bar{\mathbf{a}} \neq \mathbb{E}[\mathbf{a}]$.

Subadditive Quantity

- The averaged Dirichlet energy in finite volume captures the nature

$$\nu(U,p) := \inf_{\phi \in H_0^1(U)} \frac{1}{|U|} \int_U \frac{1}{2}(p+\nabla\phi)\cdot \mathbf{a}(p+\nabla\phi) = \frac{1}{2} p \cdot \mathbf{a}(U) p.$$

- $\nu(U,p)$ is a subadditive quantity, i.e. $U = \sqcup_{i=1}^n U_i$,

$$\nu(U,p) \leqslant \sum_{i=1}^n \frac{|U_i|}{|U|} \nu(U_i, p).$$

- We define $\bar{\mathbf{a}} := \lim_{m\to\infty} \mathbb{E}[\mathbf{a}(Q_{3^m})]$.
- Observed in Dal Maso-Modica'86 for elliptic equation, and quantitative version in Armstrong-Smart'16 and developed in Armstrong-Kuusi-Mourrat'17.

Quantitative Homogenization

The renormalization approach now applies to various models: the finite-difference equations on percolation clusters (Armstrong-Dario'18, Dario'18, Dario-G.'21), the differential forms (Dario 18), the "$\nabla\phi$" interface model (Dario'19, Armstrong-Wu'19), the Villain model (Dario-Wu'20), the Coulomb gases (Armstrong-Serfaty'19), the interacting particle system (Giunti-G.-Mourrat'22, Giunti-G.-Mourrat-Nitzschner'22).

History

Figure: Some researchers who contribute to homogenization theory: Alain Bensoussan, Jacques-Louis Lions, George Papanicolaou, Ennio De Giorgi, François Murat, Luc Tartar, Thomas Spencer, S. R. Srinivasa Varadhan, Tatsien Li, Grégoire Allaire, Marco Avellaneda, Carlos Kenig, Fanghua Lin, Zhongwen Shen, Felix Otto, Antoine Gloria, Stefan Neukamm, Scott Armstrong, Charles Smart, Jean-Christophe Mourrat, Tuomo Kuusi.

Outline

Random Conductance Model

Homogenization

Interacting Particle Systems

Main Results

Simple Symmetric Exclusion Process

- Configuration $\eta : \mathbb{Z}^d \to \{0,1\}$.
- Generator
$\mathcal{L}f(\eta) = \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{y\sim x}\eta(x)(1-\eta(y))\left(f(\eta^{x,y}) - f(\eta)\right)$ with

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & z \neq x, y; \\ \eta(y) & z = x; \\ \eta(x) & z = y. \end{cases}$$

- Stationary measure is product Bernoulli measure $\mathrm{Ber}(\alpha)^{\otimes\mathbb{Z}^d}$ with $\alpha \in (0,1)$.

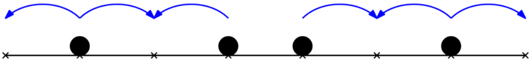


Figure: An illustration of SSEP.

Simple Symmetric Exclusion Process

- Empirical measure $\pi_t^N := N^{-d} \sum_{x \in \mathbb{Z}^d} \eta_{N^2 t}(x) \delta_{x/N}$.
- Hydrodynamic limit $(\pi_t^N)_{t \geqslant 0} \xrightarrow{N \to \infty} (\rho_t)_{t \geqslant 0}$ with

$$\partial_t \rho_t = \frac{1}{2} \Delta \rho_t.$$

- Equilibrium fluctuation
  $Y_t^N := N^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} \left(\eta_{N^2 t}(x) - \alpha\right) \delta_{x/N}$ converges to the functional Ornstein–Uhlenbeck process $(Y_t)_{t \geqslant 0}$ solving

$$\mathrm{d}Y_t = \frac{1}{2} \Delta Y_t \, \mathrm{d}t + \sqrt{\alpha(1-\alpha)} \nabla \mathrm{d}B_t.$$

Generalized Symmetric Exclusion Process

- Configuration $\eta : \mathbb{Z}^d \to \{0, 1, \cdots, \kappa\}, \kappa \geqslant 2$.
- Generator
  $\mathcal{L}f(\eta) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \sum_{y \sim x} \mathbf{1}_{\{\eta(x)>0, \eta(y)<\kappa\}} \left(f(\eta^{x,y}) - f(\eta)\right)$,
  with
  $$\eta^{x,y}(z) = \begin{cases} \eta(z) & z \neq x, y; \\ \eta(x) - 1 & z = x; \\ \eta(y) + 1 & z = y. \end{cases}$$
- Stationary measure $\mathbb{P}_\alpha = \nu_\alpha^{\otimes \mathbb{Z}^d}$ with
  $\nu_\alpha(n) = \frac{\alpha^n}{\sum_{j=0}^{\kappa} \alpha^j}, 0 \leqslant n \leqslant \kappa$.

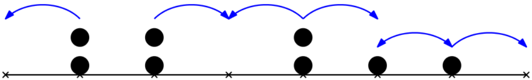


Figure: An illustration of GSEP with $\kappa = 2$.

Generalized Symmetric Exclusion Process

- Hydrodynamic limit $(\pi_t^N)_{t\geqslant 0} \xrightarrow{N\to\infty} (\rho_t)_{t\geqslant 0}$ with

$$\partial_t \rho_t = \nabla \cdot (D(\rho_t)\nabla \rho_t)\,.$$

  proved in Varadhan'90, Kipnis-Landim-Olla'94, Funaki-Uchiyama-Yau'96.

- Bulk diffusion matrix $D(\alpha) := \frac{\hat{c}(\alpha)}{2\chi(\alpha)}$, $\chi(\alpha) := \mathrm{Var}_\alpha[\eta(0)]$,

$$p\cdot \hat{c}(\alpha)p := \inf_{f\in C_0} \sum_{i=1}^{d} \mathbb{E}_\alpha \left[\mathbf{1}_{\{\eta(0)>0,\eta(e_i)<\kappa\}}(p_i + \nabla_{0,e_i}\Gamma_f(\eta))^2\right],$$

  with f local function and $\Gamma_f(\tilde{\eta}) := \sum_{x\in\mathbb{Z}^d} \tau_x f(\tilde{\eta})$.

- Observation: The bulk diffusion matrix should play the same role as the effective coefficient.

Outline

Diffusion on Continuum Configuration Space

- Particles seen as configuration $\mu_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \in \mathcal{M}_\delta(\mathbb{R}^d)$.
- Diffusion matrix $\mathbf{a}_\circ : \mathcal{M}_\delta(\mathbb{R}^d) \to \mathbb{R}^{d\times d}_{sym}$
  - locality: $\mathcal{F}_{B_1}$-measurable;
  - uniform ellipticity: $|\xi|^2 \leq \xi \cdot \mathbf{a}_\circ(\mu)\xi \leq \Lambda|\xi|^2$.
  - stationarity: $\mathbf{a}(\mu, x) := \tau_x \mathbf{a}_\circ(\mu) = \mathbf{a}_\circ(\tau_{-x}\mu)$.
- $(X_t^i)_{t\geqslant 0}$ diffuses following $\nabla \cdot \mathbf{a}(\mu_t, X_t^i)\nabla$ in $\mathbb{R}^d$, where $\mathbf{a}(\mu_t, X_t^i)$ depends on the local configuration in $B_1(X_t^i)$.
- Construction of similar processes by Albeverio, Kondratiev, Ma, Röckner 97-00 and functional inequalities by Röckner, Wang'01.

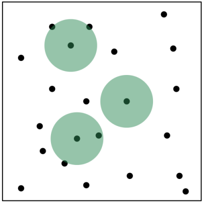


Figure: Each particle diffuses following the generator $\nabla \cdot \mathbf{a}\nabla$, where $\mathbf{a}$ depends on the position and the local configuration around the particle.

Dirichlet Energy for Particle System

- Stationary measure $\mathbb{P}_\rho$ as Poisson point process of density ρ.
- Derivative $\partial_k f(\mu, x) := \lim_{h\to 0} \frac{1}{h}(f(\mu - \delta_x + \delta_{x+he_k}) - f(\mu))$.
- Finite volume approximation

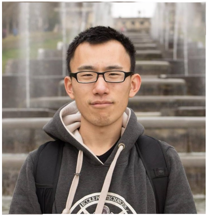
$$
\begin{aligned}
&\nu(Q_r, \rho, p) \\
&:= \inf_{\phi \in \mathscr{H}_0^1(Q_r)} \mathbb{E}_\rho\left[\frac{1}{\rho|Q_r|}\int_{Q_r} \frac{1}{2}(p + \nabla\phi)\cdot \mathbf{a}(p+\nabla\phi)\, d\mu\right] \\
&= \frac{1}{2} p \cdot \bar{\mathbf{a}}(Q_r, \rho) p.
\end{aligned}
$$

Main Result

Theorem (Giunti-G.-Mourrat, AoP 2022)

The limit $\bar{\mathbf{a}}(\rho) := \lim_{r\to\infty} \bar{\mathbf{a}}(Q_r, \rho)$ *exists, coincides with the definition of bulk diffusion matrix, and we have*

$$|\bar{\mathbf{a}}(Q_r, \rho) - \bar{\mathbf{a}}(\rho)| \leqslant C r^{-\alpha}.$$


A joint work with Arianna Giunti and Jean-Christophe Mourrat.

Proof

1. Renormalization approach.
2. Good function space on configuration space.
3. Modified Caccioppoli inequality.

Proof: Subadditive Quantity

- Function space $\mathscr{H}^1(U)$ has finite Dirichlet energy in U, while $\mathscr{H}_0^1(U)$ is $\mathcal{F}_K$-measurable for $K \subset U$.
- Dual quantity

$$\begin{aligned}
&\nu_*(Q_r, \rho, q) \\
&:= \inf_{v \in \mathscr{H}^1(Q_r)} \mathbb{E}_\rho \left[\frac{1}{\rho |Q_r|} \int_{Q_r} \left(-\frac{1}{2} \nabla v \cdot \mathbf{a} \nabla v + q \cdot \nabla v \right) d\mu \right] \\
&= \frac{1}{2} q \cdot \mathbf{a}_*^{-1}(U) q.
\end{aligned}$$

- ν, ν_* are all subadditive quantities.
- Some similar qualitative result for SSEP is shown by Landim-Olla-Varadhan'02.

$\mathbb{R}^d$-Classical Caccioppoli Inequality

- The classical Caccioppoli inequality for $\mathbf{a}$-harmonic function on $\mathbb{R}^d$

$$\int_{Q_r} |\nabla \tilde{u}|^2 \leqslant \frac{C}{r^2} \int_{Q_{3r}} |\tilde{u}|^2.$$

- Elementary but an important step to establish the elliptic regularity theory.

Modified Caccioppoli inequality

- $\mathbf{a}$-harmonic function on particle system

$$u \in \mathcal{A}(U) \iff \forall v \in \mathscr{H}_0^1(U), \mathbb{E}_\rho \left[\int_U \nabla u \cdot \mathbf{a} \nabla v \, \mathrm{d}\mu \right] = 0.$$

- There exists $\theta(d, \Lambda) \in (0,1)$, such that for every $u \in \mathcal{A}(Q_{3r})$

$$\mathbb{E}_\rho \left[\frac{1}{\rho |Q_r|} \int_{Q_r} \nabla (\mathsf{A}_{r+2} u) \cdot \mathbf{a} \nabla (\mathsf{A}_{r+2} u) \, \mathrm{d}\mu \right] \leqslant \frac{C}{r^2 \rho |Q_{3r}|} \mathbb{E}_\rho [u^2] + \theta \mathbb{E}_\rho \left[\frac{1}{\rho |Q_{3r}|} \int_{Q_{3r}} \nabla u \cdot \mathbf{a} \nabla u \, \mathrm{d}\mu \right].$$

- $\mathsf{A}_s u := \mathbb{E}_\rho [u | \mathcal{F}_{\overline{Q}_s}]$.
- Proof = Classical Caccioppoli inequality (L^2-martingale structure) + Widman's hole-filling technique + iterations.

Further Discussions

- Regularity of the mapping $\rho \mapsto \bar{\mathbf{a}}(\rho)$? C^∞ is proved in Giunti-G.-Nitzschner-Mourrat'22, but what about the analyticity ?
- Hydrodynamic limit convergence rate.
- Homogenization for GSEP ? And more singular interactions ? (In preparation with Tadahisa Funaki and Han Wang.)
- Application to other particle systems and other problems.

Thanks for your attention.